

Maps of Mori Dream Spaces in Cox coordinates

Part I: existence of descriptions

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Abstract

Any rational map between affine spaces, projective spaces or toric varieties can be described in terms of their affine, homogeneous, or Cox coordinates. We show an analogous statement in the setting of Mori Dream Spaces. More precisely (in the case of regular maps) we show there exists a finite extension of the Cox ring of the source, such that the regular map lifts to a morphism from the Cox ring of the target to the finite extension. Moreover the extension only involves roots of homogeneous elements. Such a description of the map can be applied in practical computations.

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1 Introduction

By definition, an algebraic morphism of two affine varieties $\varphi: X \rightarrow Y$ is a geometric interpretation of an algebra homomorphism $\varphi^*: B \rightarrow A$ of their

affine coordinate rings. Here $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. If $X = \mathbb{P}^m$ and $Y = \mathbb{P}^n$ instead, and $A \simeq \mathbb{k}[x_0, \dots, x_m]$ and $B \simeq \mathbb{k}[y_0, \dots, y_n]$ are their homogeneous coordinate rings, then any algebraic morphism $\varphi: \mathbb{P}^m \rightarrow \mathbb{P}^n$ is determined a homomorphism $B \rightarrow A$ satisfying the usual homogeneity and base point freeness conditions. Rational maps between affine varieties or projective spaces have similar interpretations in terms of the fields of fractions of coordinate rings.

Furthermore, an analogous coordinatewise description applies also to a rational map $\varphi: X \dashrightarrow Y$ between toric varieties X and Y [BB13], [Cox95]. In this case the underlying algebra involves the *Cox rings* of toric varieties, also called their *homogeneous coordinate rings*, or *total coordinate rings*. The major difference between previous settings is that we need to extend the Cox ring of X (or its field of fractions) by several roots of homogeneous polynomials. The number and the order of the roots is essentially bounded by the singularities of Y .

In the present article we extend this analogy to arbitrary *Mori Dream Spaces*. Suppose a normal algebraic variety X over an algebraically closed field \mathbb{k} of characteristic 0 has a finitely generated divisor class group $\operatorname{Cl}(X)$. Then we define its Cox ring to be:

$$S[X] := \bigoplus_{[D] \in \operatorname{Cl}(X)} H^0(\mathcal{O}(D)).$$

This \mathbb{k} -vector space has a well defined ring structure (see [ADHL15, §1.4]), which makes it a multigraded ring with a grading by $\operatorname{Cl}(X)$. We say X is a *Mori Dream Space* (or *MDS*), if $S[X]$ is a finitely generated \mathbb{k} -algebra. In this situation there is a natural quotient map $\pi_X: \operatorname{Spec} S[X] \dashrightarrow X$ by the action of the quasitorus $G_X = \operatorname{Spec} \mathbb{k}[\operatorname{Cl}(X)] = \operatorname{Hom}(\operatorname{Cl}(X), \mathbb{k}^*)$.

Affine and projective spaces and normal toric varieties are Mori Dream Spaces. In these cases, the Cox ring is always a polynomial ring, but the grading vary. In fact, the property that $S[X]$ is a polynomial ring characterises toric varieties, see [KW15] for a recent treatment of this characterisation. Further examples of Mori Dream Spaces include projectivisations of rank two toric vector bundles [Gonz12], del Pezzo surfaces [BP04] and some K3 surfaces [AHL10]. Every log Fano variety is a Mori Dream Space by [BCHM10], in particular the moduli spaces of pointed rational curves $\overline{M}_{0,n}$ have finitely generated Cox rings for $n \leq 6$. Minimal resolutions of surface quotient singularities are interesting non-projective examples of Mori Dream Spaces [DB13].

In brief, our main result is the following.

Theorem 1.1. *Suppose X and Y are Mori Dream Spaces, and $\varphi: X \dashrightarrow Y$ is a rational map. Then there exists a description of φ in terms of Cox coordinates, that is a multi-valued map*

$$\Phi: \operatorname{Spec} S[X] \multimap \operatorname{Spec} S[Y]$$

such that for all points $x \in X$ and ξ such that $\pi_X(\xi) = x$ and φ is regular at x , the composition $\pi_Y(\Phi(\xi))$ is a single point $\varphi(x) \in Y$.

The precise definition of *multi-valued map* is presented in Section 3. It follows the convention of [BB13, §3.2], where the same notion is used in the setup of toric varieties, that is the special case $\operatorname{Spec} S[X] = \mathbb{k}^m$ and $\operatorname{Spec} S[Y] = \mathbb{k}^n$. The theorem is effective in the sense, that the proof shows how to construct the description.

Similar statement for regular maps between \mathbb{Q} -factorial Mori Dream Spaces has been independently obtained by Andreas Hochenegger and Elena Martinengo [HM16]. Their approach is to use the language of Mori Dream stacks [HM15]. They use the technique of root constructions, which is parallel to the multi-valued maps in this article.

Theorem 1.1 is proved in several steps, concluding in Theorem 8.3 in Section 8. It is a direct generalisation of the results of [BB13]. Admittedly, a large part of this article adapts [BB13, Sections 3 and 4] to the more general setting. We heavily exploit the theory of Cox rings described in the book [ADHL15] in order to make the generalisation possible.

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2 Preliminaries

Fix an algebraically closed base field \mathbb{k} of characteristic 0. Suppose X is a normal algebraic variety over \mathbb{k} with a finitely generated class group $\operatorname{Cl}(X)$. We also assume that there are no non-constant global invertible functions on X , that is $H^0(X, \mathcal{O}_X^*) = \mathbb{k}^*$. Then \mathcal{S}_X , the *Cox sheaf* of X , is the \mathcal{O}_X -algebra defined as

$$\mathcal{S}_X = \bigoplus_{[D] \in \operatorname{Cl}(X)} \mathcal{O}_X(D),$$

and the *Cox ring* of X is the ring of global sections $S[X] = H^0(X, \mathcal{S}_X)$. The construction of the multiplication structures in \mathcal{S}_X and $S[X]$ is slightly

delicate, especially in the case when $\text{Cl}(X)$ is not torsion free, see [ADHL15, §1.4.2] for the details.

Assume X is a *Mori Dream Space*, that is $S[X]$ is finitely generated. Then $S[X]$ is a domain by [ADHL15, Thm 1.5.1.1].

We begin with listing the notation that will be used throughout the article.

- The field of fractions of the Cox ring of X will be denoted by $S(X)$. The contrast between the ring $S[X]$ and field $S(X)$ is analogous to the standard notation for polynomial ring $\mathbb{k}[x_1, \dots, x_m]$ and rational function field $\mathbb{k}(x_1, \dots, x_m)$. Elements of $S[X]$ will be called *regular sections* on X , elements of $S(X)$ will be called *rational sections* on X .
- $K(X)$ is the field of rational functions on X . Elements of $K(X)$ will be called *rational functions* on X .
- We fix an algebraic closure of the field $S(X)$ and we denote it $\overline{S(X)}$.
- Since $S[X]$ is a graded ring, we can define the degree 0 subfield of $S(X)$:

$$S(X)^0 = \left\{ \frac{f}{g} \in S(X) \mid f, g \in S[X], \deg f = \deg g \right\}.$$

Note that $S(X)^0 = K(X)$ by Proposition 2.1.

- We say $q \in S(X)$ is homogeneous if it is a ratio of homogeneous elements of $S[X]$, i.e. $q = \frac{f}{g}$ with f, g homogeneous. Nevertheless note that $S(X)$ is not itself a graded ring.
- By G_X we denote $\text{Spec } \mathbb{k}[\text{Cl}(X)]$, the *characteristic (quasi)torus*, whose character group is $\text{Cl}(X)$.
- Let $\widehat{X} := \text{Spec}_X \mathcal{S}_X$, the relative spectrum of the sheaf of algebras \mathcal{S}_X . The variety \widehat{X} is called the *characteristic space* of X .
- Let $\overline{X} = \text{Spec } S[X]$ be the *total coordinate space* of X (or a *Cox cover*).
- Both \widehat{X} and \overline{X} admit the action of G_X determined by the grading of \mathcal{S}_X and $S[X]$. The natural map $\widehat{X} \rightarrow \overline{X}$ is a G_X -equivariant open embedding [ADHL15, Construction 1.6.3.1]. The complement $\text{Irrel}(X) := \overline{X} \setminus \widehat{X}$ is called the *irrelevant locus*.
- The *irrelevant ideal* B_X is the homogeneous ideal in $S[X]$ defining $\text{Irrel}(X) \subset \overline{X}$.

- Let $\pi_X: \overline{X} \dashrightarrow X$ be the canonical rational map. Note that $\pi_X|_{\widehat{X}}$ is regular, surjective and it is the good quotient of \widehat{X} by G_X [ADHL15, Construction 1.6.1.3].
- Suppose $A \subset X$ is a closed subscheme. Then the *ideal* of A is the homogeneous ideal $I(A) \subset S[X]$ generated by all sections vanishing on A .

We will freely use the following equivalence:

Proposition 2.1. *The rational function field $K(X)$ of a Mori Dream Space X is naturally isomorphic to the field $S(X)^0$ of degree 0 rational sections.*

Proof. Suppose f and g are homogeneous elements of $S[X]$ of the same degree. Then $f, g \in H^0(\mathcal{O}_X(D))$ for a divisor D on X . By definition

$$H^0(\mathcal{O}_X(D)) = \{q \in K(X) \mid D + (q) \geq 0\}.$$

Suppose f corresponds to $q_1 \in K(X)$ and g corresponds to $q_2 \in K(X)$ in this definition. Then

$$S(X)^0 \ni \frac{f}{g} = \frac{q_1}{q_2} \in K(X),$$

and this correspondence naturally defines a homomorphism $S(X)^0 \rightarrow K(X)$. It remains to verify that this map is surjective.

Take a non-zero $q \in K(X)$. It defines a principal divisor $(q) \subset X$. Write $(q) = D^+ - D^-$, where both D^+ and D^- are effective linearly equivalent divisors. Hence there exist homogeneous $f, g \in S[X]$ of the same degree with $D^+ = (f)$ and $D^- = (g)$. Thus $f/g \in K(X)$ and $\frac{q}{f/g}$ is a rational function defining a trivial divisor, hence q is a rescaling of f/g by a globally invertible regular function, which must be constant by our assumptions. \square

2.1 Gradings and fields

Let M be a finitely generated abelian group, and $S = \bigoplus_{m \in M} S^m$ an M -graded domain. Let \mathbb{F} be the field of fractions of S , so that we have $S \subset \mathbb{F}$. Consider the subfield of degree 0 elements

$$\mathbb{F}^0 = \left\{ \frac{f}{g} : f, g \in S, \deg f - \deg g = 0 \right\}.$$

If $A \subset \mathbb{F}$ is a subset, then by $\mathbb{F}^0(A)$ we denote the smallest subfield of \mathbb{F} containing \mathbb{F}^0 and A , i.e. the field generated over \mathbb{F}^0 by A .

Given a subgroup M' of M we may also consider a coarser grading on S , namely by the finitely generated group M/M' : If $[m] \in M/M'$, then $S_{[m]} = \bigoplus_{m' \in M'} S_{m+m'}$ with respect to the coarser grading. We define the subfield of degree zero elements with respect to the coarser grading:

$$\mathbb{F}^{M'} = \left\{ \frac{f}{g} : f, g \in S_{[m]} \text{ for some } [m] \in M/M' \text{ common to } f \text{ and } g \right\}.$$

Lemma 2.2. *Suppose $A \subset S$ is a subset of the ring consisting only of homogeneous elements (in particular, A may be infinite). Let $\langle \deg A \rangle \subset M$ be the subgroup of M generated by the degrees of elements of A . (For consistence, we assume $\deg 0 = 0$.) Then*

$$\mathbb{F}^0(A) = \mathbb{F}^{\langle \deg A \rangle}.$$

In particular, there is a finite subset $A' \subset A$ such that $\mathbb{F}^0(A) = \mathbb{F}^0(A')$.

Proof. We always have $\mathbb{F}^0 \subset \mathbb{F}^{\langle \deg A \rangle}$ and $A \subset \mathbb{F}^{\langle \deg A \rangle}$, so:

$$\mathbb{F}^0(A) \subset \mathbb{F}^{\langle \deg A \rangle}. \quad (2.3)$$

To show the reverse inclusion $\mathbb{F}^0(A) \supset \mathbb{F}^{\langle \deg A \rangle}$, we first suppose the set A consists of a single homogeneous element $A = \{a\}$. If $a = 0$, there is nothing to prove. Otherwise, pick $q \in \mathbb{F}^{\langle \deg A \rangle}$, i.e.

$$q = \frac{f_i + f_{i+1} + \cdots + f_j}{g_k + g_{k+1} + \cdots + g_l}$$

where $i, j, k, l \in \mathbb{Z}$ and $f_n, g_n \in S_{n \deg a}$. In particular, $\frac{f_n}{a^n} \in \mathbb{F}^0$ and analogously $\frac{g_n}{a^n} \in \mathbb{F}^0$. Thus:

$$q = \frac{\frac{f_i}{a^i} a^i + \cdots + \frac{f_j}{a^j} a^j}{\frac{g_k}{a^k} a^k + \cdots + \frac{g_l}{a^l} a^l}$$

which expresses q in terms of a and \mathbb{F}^0 . Thus $\mathbb{F}^{\langle \deg a \rangle} \subset \mathbb{F}^0(\{a\})$.

Now assume A is finite. We can show the statement inductively, by adding the generators one by one and applying the initial case (with A equal to a single element). More precisely, let $A = A' \cup \{a\}$. By the inductive assumption $\mathbb{F}^0(A') = \mathbb{F}^{\langle \deg(A') \rangle}$ and we have

$$\mathbb{F}^0(A) = (\mathbb{F}^0(A'))(\{a\}) = \mathbb{F}^{\langle \deg(A') \rangle}(\{a\}).$$

Considering the coarser grading by $M/\langle \deg(A') \rangle$ the field $\mathbb{F}^{\langle \deg(A') \rangle}$ is really a subfield of degree $[0]$ elements (with respect to the coarser grading). Thus we apply the initial case to obtain $\mathbb{F}^{\langle \deg(A') \rangle}(\{a\}) = \mathbb{F}^{\langle \deg(A) \rangle}$.

Finally, suppose A is arbitrary. Since M is finitely generated and abelian, also $\langle \deg A \rangle$, which is a subgroup of M , is finitely generated. Pick a finite subset $A' \subset A$, such that $\langle \deg A' \rangle = \langle \deg A \rangle$. Using this equality and the case when A is finite we obtain

$$\mathbb{F}^0(A) \stackrel{(2.3)}{\subset} \mathbb{F}^{\langle \deg A \rangle} = \mathbb{F}^{\langle \deg A' \rangle} = \mathbb{F}^0(A') \stackrel{A' \subset A}{\subset} \mathbb{F}^0(A).$$

Thus all the inclusions above are equalities, i.e. $\mathbb{F}^0(A) = \mathbb{F}^{\deg(A)} = \mathbb{F}^0(A')$, concluding the proof. \square

2.2 Maps of fields with kernel

Given two algebraic varieties X and Y over an algebraically closed field \mathbb{k} we consider a rational map $\varphi: X \dashrightarrow Y$. If φ is dominant, then such map is uniquely determined by a homomorphism of fields $\varphi^*: K(Y) \rightarrow K(X)$. In this section we are interested in the algebraic interpretation of the case when $\varphi: X \dashrightarrow Y$ is not necessarily dominant, compare to [BB13, Prop. 2.14(i),(ii)].

One approach is to replace Y with the closure of the image $Z := \overline{\varphi(X)}$. Then $\varphi: X \dashrightarrow Z$ is a dominant and there is a homomorphism $\varphi^*: K(Z) \rightarrow K(X)$. However, this does not help very much to describe, for example, closed embeddings. Here we are seeking a uniform description in terms of $K(Y)$, as we are going to extend this description to Cox ring $S[Y]$. For this purpose we define the notion of a *map of fields with kernel*.

Definition 2.4. Suppose \mathbb{F} and \mathbb{G} are fields. A *map of fields with kernel* (denoted $\Phi^*: \mathbb{F} \dashrightarrow \mathbb{G}$) is a ring homomorphism $\Phi^*: R \rightarrow \mathbb{G}$, where:

- $R \subset \mathbb{F}$ is a subring which generates \mathbb{F} (i.e. \mathbb{F} is the smallest subfield of \mathbb{F} containing R),
- if $f \in R$ and $f^{-1} \in \mathbb{F} \setminus R$, then $\Phi^*(f) = 0$.

Thus if $\varphi: X \dashrightarrow Y$ is a rational map, then let $R \subset K(Y)$ be the local ring of the scheme-theoretic point, whose closure is $\overline{\varphi(X)}$, or equivalently, the set of those rational functions on Y , whose set of poles does not contain $\varphi(X)$. Clearly R generates \mathbb{F} , and $\varphi^*: R \rightarrow K(X)$ is a well defined homomorphism. If $f \in R$ and $f^{-1} \in \mathbb{F} \setminus R$, then $\varphi(X)$ is contained in the set of poles of f^{-1} and but is not contained in the set of poles of f , hence f must be 0 along $\overline{\varphi(X)}$. Thus $\varphi^*: K(Y) \dashrightarrow K(X)$ is a map of fields with kernel.

Lemma 2.5. Let \mathbb{F}, \mathbb{G} be fields, $R' \subset \mathbb{F}$ be a subring generating \mathbb{F} , and let $\Phi^*: R' \rightarrow \mathbb{G}$ be a ring homomorphism. Then there is a unique map of fields with kernel $\mathbb{F} \dashrightarrow \mathbb{G}$ extending Φ^* .

Proof. Consider the kernel of $\Phi^*: R' \rightarrow \mathbb{G}$ and the localisation $R := (R')_{\ker \Phi^*}$ of R' in this ideal. Then Φ^* extends uniquely to R and R generates \mathbb{F} . If $f \in R$ and $f^{-1} \notin R$, then $f \in \ker \Phi^*$, hence $\Phi^* f = 0$ and $\Phi^*: \mathbb{F} \dashrightarrow \mathbb{G}$ is a map of fields with kernel. \square

3 Multivalued sections and maps

Suppose X and Y are Mori Dream Spaces, and consider their total coordinate spaces $\overline{X} = \operatorname{Spec} S[X]$ and $\overline{Y} = \operatorname{Spec} S[Y]$.

Definition 3.1. A *homogeneous multi-valued section* on X is an element $\gamma \in \overline{S(X)}$ of the algebraic closure of the fraction field $S(X)$, such that $\gamma^r \in S(X)$ is a homogeneous rational section on X for some positive integer r .

We always pick minimal r , such that $\gamma^r = q \in S(X)$, and then we write $\gamma = \sqrt[r]{q}$. Then a value of γ at a point $\xi \in \overline{X}$ is $\gamma(\xi) = \left\{ \sqrt[r]{q(\xi)} \right\} \subset \mathbb{k}$, i.e. the set of all r -th roots of $q(\xi)$ (assuming ξ is not a pole of q).

Proposition 3.2. A homogeneous multi-valued section $\gamma \in \overline{S(X)}$ is in $S(X)$ if and only if $\gamma(\xi)$ has exactly one element for a general $\xi \in \overline{X}$. \square

Proposition 3.3. If V is a \mathbb{k} -vector space and $i: V \rightarrow \overline{S(X)}$ is a \mathbb{k} -linear map whose image consists of only homogeneous multi-valued sections, then there exists a homogeneous multi-valued section $\gamma \in \overline{S(X)}$ and a \mathbb{k} -linear map $j: V \rightarrow S(X)$ whose image consists of homogeneous elements of a constant degree, and $i(v) = j(v) \cdot \gamma$ for all $v \in V$.

See [BB13, Cor. 2.20 and Prop. 3.6]; the main ingredient of the proof is [BB13, Cor. 2.20], which is valid over any base field, the only assumption is that $\mathbb{F} = S(X)$ contains all roots of unity. On the other hand, the statement of [BB13, Prop. 3.6] is analogous to Proposition 3.3, but restricted to toric varieties (i.e. $S(X)$ is a field of rational functions). However this assumption is never exploited in the simple argument.

A regular map between affine varieties is merely a geometric interpretation of a homomorphism of respective coordinate rings. In the same spirit we want to give a geometric interpretation to a more general homomorphism, which is composed of homogeneous multi-valued sections.

For this purpose, fix y_1, \dots, y_N , a finite number of homogeneous generators of the Cox ring $S[Y]$. We will see later, in Section 4, that the particular choice of the generators does not matter from the point of view of our needs.

Definition 3.4. A *multi-valued map* $\Phi: \overline{X} \multimap \overline{Y}$ is a geometric interpretation of a ring homomorphism $\Phi^*: S[Y] \rightarrow \overline{S(X)}$, which maps the generators y_i to homogeneous multi-valued sections of X . That is, for all i , there exists r_i , such that $\Phi^*(y_i^{r_i}) \in S(X)$ is a homogeneous rational section on X .

The purely algebraic definition of $\Phi: \overline{X} \multimap \overline{Y}$ has now immediate geometric consequences. First, we define $\text{Reg } \Phi \subset \overline{X}$ the *regular locus* of Φ , to be the open dense subset, complementary to the zero sets of denominators appearing in $(\Phi^* y_i)^{r_i}$. That is, we write $\Phi^* y_i = \sqrt[r_i]{\frac{f_i}{g_i}}$ for homogeneous regular sections $f_i, g_i \in S[X]$, and we assume f_i and g_i have no common homogeneous factors. This is possible by the graded factoriality of the Cox ring, see [ADHL15, Theorem 1.5.3.7]. Then:

$$\text{Reg } \Phi := \{\xi \in \overline{X} \mid \forall_i g_i(\xi) \neq 0\} = \overline{X} \setminus Z(g_1 \cdots g_N).$$

Corollary 3.5. *Reg Φ is the complement of a G_X -invariant divisor in \overline{X} and it is affine.*

Suppose $\Phi: \overline{X} \multimap \overline{Y}$ is a multi-valued map. Since $S[Y]$ generates $S(Y)$, the homomorphism Φ^* determines a unique map of fields with kernel $\Phi^*: S(Y) \dashrightarrow \overline{S(X)}$ by Lemma 2.5. Specifically, pick $q = \frac{f}{g} \in S(Y)$ with $f, g \in S[Y]$. Assume $\Phi^* g \neq 0$, then $\Phi^* q := \frac{\Phi^* f}{\Phi^* g}$ is well defined, and thus Φ^* naturally extends to:

$$\Phi^*: R \rightarrow \overline{S(X)}, \text{ where } R \text{ is the subring } R = \left\{ q = \frac{f}{g} \in S(Y) \mid \Phi^* g \neq 0 \right\}. \quad (3.6)$$

Remark 3.7. In particular, R contains a subring $R^0 \subset R$ of quotients of degree 0, hence $R^0 \subset K(Y)$.

Let $\Gamma(\Phi)$ be the subring of $\overline{S(X)}$ generated by $S[X][g^{-1}]$, and $\Phi^*(S[Y])$, where $g = g_1 \cdots g_N$, so that $\text{Reg } \Phi = \text{Spec } S[X][g^{-1}]$. Then we have the following algebraic morphisms:

$$\overline{X} \supset \text{Reg } \Phi \xleftarrow{p_\Phi} \text{Spec } \Gamma(\Phi) \xrightarrow{q_\Phi} \text{Spec } S[Y].$$

The first map p_Φ is the morphism of affine varieties corresponding to the inclusion $p_\Phi^*: S[X][g^{-1}] \rightarrow \Gamma(\Phi)$. The second map q_Φ corresponds to $q_\Phi^* = \Phi^*: S[Y] \rightarrow \Gamma(\Phi)$. Using these two maps we define the (set-theoretic) image and preimage under a multi-valued map:

Definition 3.8. Suppose $Z \subset \overline{X}$ and $W \subset \overline{Y}$ are two subsets. Then define:

$$\Phi(Z) := q_\Phi(p_\Phi^{-1}(Z)) \quad \text{and} \quad \Phi^{-1}(W) := p_\Phi(q_\Phi^{-1}(W)).$$

In particular, the following are immediate consequences of the definition:

Proposition 3.9. *In the setting of Definition 3.8:*

- (i) $\Phi(Z) = \Phi(Z \cap \text{Reg } \Phi)$
- (ii) $\Phi^{-1}(W) \subset \text{Reg } \Phi$.
- (iii) $\Phi(Z) = \emptyset$ if and only if $Z \subset \overline{X} \setminus \text{Reg } \Phi$, i.e. $\text{Reg } \Phi$ is the set of points, where the image under Φ is defined.
- (iv) For $g \in S[Y]$ we have $\Phi^*g = 0$ if and only if $\Phi(\overline{X}) \subset (g = 0)$.
- (v) Let $I = \ker \Phi^*$. Then the ring R from Equation (3.6) is equal to the (non-homogeneous) localisation $S[Y]_I$.

Observe that $p_\Phi: \text{Spec } \Gamma(\Phi) \rightarrow \text{Reg } \Phi$ is a finite morphism. In particular, p_Φ is closed. Thus (since q_Φ is continuous):

Proposition 3.10. *For a closed subset $W \subset \overline{Y}$ the preimage $\Phi^{-1}(W)$ is closed in $\text{Reg } \Phi$.*

Remark 3.11. It is also possible to show that the preimage (under a multi-valued map Φ) of an open subset is open, thus Φ has very much of properties of continuous mappings. This statement requires a more refined definition of Γ_Φ , and we postpone it until the second part of these series of papers. See also [BB13, §3.3, §3.4, Prop. 3.17].

4 Homogeneity conditions

Analogously to [BB13, Prop. 4.6], we prove the equivalence of several conditions, jointly referred to as *homogeneity conditions*.

Recall the setup: X and Y are Mori Dream Spaces, with total coordinate spaces $\overline{X} = \text{Spec } S[X]$ and $\overline{Y} = \text{Spec } S[Y]$. We pick some homogeneous generators y_1, \dots, y_N of $S[Y]$.

Proposition 4.1. *Suppose $\Phi: \overline{X} \dashrightarrow \overline{Y}$ is a multi-valued map and consider the set*

$$L = \{y_i \mid i \in \{1, \dots, N\} \text{ and } \Phi^*y_i \neq 0\}$$

of the fixed generators the Cox ring $S[Y]$ that pull back nontrivially under Φ . Further let $\mathbb{F} \subset S(Y)$ be the subfield generated by L , and let $\mathbb{F}^0 := \mathbb{F} \cap S(Y)^0 \subset K(Y)$. The following conditions are equivalent:

- (A1) If $q \in S(Y)$ is homogeneous and Φ^*q is defined (i.e. the pullback of the denominator is non-zero), then Φ^*q is a homogeneous multi-valued section on X .
- (A2) If $q \in K(Y)$ and Φ^*q is defined (i.e. $q \in R^0$ in the sense of Remark 3.7), then $\Phi^*q \in K(X)$.
- (A3) There exist l_1, \dots, l_k , elements of the field \mathbb{F}^0 , generating it as a field extension of \mathbb{k} such that Φ^*l_i are elements of $S(X)^0 = K(X)$.
- (A4) Φ maps G_X -orbits into G_Y -orbits. More precisely, for all $\xi, \xi' \in \text{Reg } \Phi$ with $\xi' \in G_X \cdot \xi$, if $\eta \in \Phi(\xi)$ and $\eta' \in \Phi(\xi')$ then $\eta' \in G_Y \cdot \eta$.
- (A4') Φ maps G_X -orbits meeting an open dense subset of X into G_Y -orbits. More precisely, there exists an open dense subset $U \subset \text{Reg } \Phi$, such that for all $\xi, \xi' \in U$ with $\xi' \in G_X \cdot \xi$, if $\eta \in \Phi(\xi)$ and $\eta' \in \Phi(\xi')$ then $\eta' \in G_Y \cdot \eta$.

Proof. Suppose (A1) holds for Φ . Let $R^0 \subset S(Y)$ be the \mathbb{k} -vector subspace of homogeneous sections of degree 0 for which the pull-back by Φ is defined as in Remark 3.7. Denote the restriction of Φ^* to R^0 by $i: R^0 \rightarrow \overline{S(X)}$. Since $i(1) = 1$ is rational and has degree 0, Proposition 3.3 implies that all elements of $i(R^0)$ are rational and of degree 0. Therefore (A2) holds for Φ .

Suppose (A2) holds. Since $\mathbb{F}^0 \subset S(Y)^0$ (in fact $\mathbb{F}^0 \subset R^0$), any generating set l_1, \dots, l_k of \mathbb{F}^0 satisfies (A3).

Suppose (A3) holds for Φ ; we show that (A1) holds. Let $q \in S(Y)$ be any homogeneous rational section. Write

$$q = \frac{\mu_1 + \dots + \mu_\alpha}{\nu_1 + \dots + \nu_\beta},$$

where the $\mu_i, \nu_j \in S[Y]$ are monomial expressions in the generators y_k (i.e. simply products of powers of the generators y_1, \dots, y_N with non-negative exponents) with $\deg \mu_i = d_1$ and $\deg \nu_j = d_2$ for all i and j . Assume that $\Phi^*(\nu_1 + \dots + \nu_\beta) \neq 0$, so Φ^*q is defined.

We have to show Φ^*q is a homogeneous multi-valued section. If the expression of μ_i as the product of the generators contains y_k , such that $\Phi^*y_k = 0$, then $\Phi^*\mu_i = 0$. Let $q' = \frac{\mu_1 + \dots + \mu_{i-1} + \mu_{i+1} + \dots + \mu_\alpha}{\nu_1 + \dots + \nu_\beta}$. Then $\Phi^*(q') = \Phi^*(q)$, hence we can replace q with q' . Repeating the operation if necessary (also for the denominator), we may assume that μ_i and ν_j are all monomial expressions in the generators from L only, i.e. $q \in \mathbb{F}$.

Since μ_{i_1}/μ_{i_2} is homogeneous of degree 0, thus $\mu_{i_1}/\mu_{i_2} \in \mathbb{F}^0$, i.e. it is expressible in terms of the generators l_i . Thus, using the assumption of

(A3), the pull-back $\Phi^*(\mu_{i_1}/\mu_{i_2})$ is a non-zero homogeneous degree 0 rational section in $S(X)$.

Each $\Phi^*\mu_i$ is a homogeneous multi-valued section (it is a product of homogeneous multi-valued sections). In particular, for every i ,

$$\Phi^*(\mu_i) = f_i \cdot \gamma$$

where $\gamma \in \overline{S(X)}$ is a fixed homogeneous multi-valued section (independent of i) and $f_i \in K(X)$. So

$$\Phi^*(\mu_1 + \cdots + \mu_\alpha) = (f_1 + \cdots + f_\alpha)\gamma.$$

Similarly, $\Phi^*(\nu_1 + \cdots + \nu_\beta) = (g_1 + \cdots + g_\beta)\delta \neq 0$, for some $\delta \in \overline{S(X)}$ and $g_j \in K(X)$. So

$$\Phi^*(q) = h \cdot \varepsilon$$

where $\varepsilon = \gamma/\delta \in \overline{S(X)}$ is homogeneous and $h = (\sum f_i)/(\sum g_j) \in K(X)$. So $\Phi^*(q)$ is a homogeneous multi-valued section and (A1) holds.

It remains to prove the implications (A1) \implies (A4) \implies (A4') \implies (A2).

Suppose (A1) holds. Let $\xi \in \text{Reg } \Phi$ and consider $G_X \cdot \xi$. The claim of (A4) is that $\Phi(G_X \cdot \xi)$ is contained in one G_Y orbit. Let $I \subset S[Y]$ be the homogeneous ideal generated by:

$$I := \langle f \in S[Y] \mid f \text{ is homogeneous and } (\Phi^*f)(\xi) = 0 \rangle$$

If $f \in I$, then $\Phi^*f(\xi') = 0$ for all ξ' in the orbit $G_X \cdot \xi$. Thus $\Phi(G_X \cdot \xi)$ is contained in the $(G_Y$ -invariant) zero locus $T := Z(I) \subset \overline{Y}$.

We claim that for any $\eta' \in \Phi(G_X \cdot \xi)$ the orbit $G_Y \cdot \eta'$ is dense in T . Suppose on the contrary, that $\overline{G_Y \cdot \eta'} \subsetneq T$, and thus there exists a homogeneous $f \in S[Y]$, which is not in I , such that $f(\eta') = 0$. Let $\xi' \in G_X \cdot \xi$ be such that $\eta' \in \Phi(\xi')$. Then $\Phi^*f(\xi') = f(\eta') = 0$. But Φ^*f is homogeneous, hence also $\Phi^*f(\xi) = 0$, a contradiction, since we assumed $f \notin I$.

A dense orbit, if exists, is unique (even if T is not irreducible!). Hence by above, the action of G_Y on T has a dense orbit and $\Phi(G_X \cdot \xi)$ is contained in this orbit. This completes the proof of (A4).

If (A4) holds, then clearly (A4') holds too.

Finally, suppose (A4') holds. Let $q \in K(Y)$ be such that Φ^*q is defined. Suppose $\xi \in U$ is general. The possible values taken by Φ^*q at ξ are simply those values taken by q at the points of the image set $\Phi(\xi)$. Setting $\xi' = \xi$ in (A4') shows that $\Phi(\xi)$ is contained in a single G_Y -orbit, and so $\Phi^*q(\xi) = \{q(\eta) \mid \eta \in \Phi(\xi)\}$ is a single number. Therefore $\Phi^*q \in S(X)$ by Proposition 3.2. For arbitrary ξ, ξ' as in (A4'),

$$(\Phi^*q)(\xi) = q(\Phi(\xi)) = q(\Phi(\xi')) = (\Phi^*q)(\xi')$$

since q is constant on G_Y -orbits. That is, Φ^*q is constant on a general G_X orbit, so Φ^*q is G_X -invariant. This proves (A2). \square

Note that by the intrinsic nature of Condition (A1) also all other Conditions (A2), (A3), (A4), (A4') are independent of the choice of the generators y_i of $S[Y]$.

5 Relevance condition

We define and briefly analyse the relevance condition, analogous to [BB13, Prop. 4.7].

Definition 5.1. Let $\Phi: \bar{X} \multimap \bar{Y}$ be a multi-valued map. We say that Φ satisfies the *relevance condition*, or Φ is *relevant*, if

(B1) the image of Φ is not contained in the irrelevant locus of Y ,

or equivalently,

(B2) $\ker \Phi^*$ does not contain the irrelevant ideal B_Y of Y .

Lemma 5.2. *In the setting as above, let the ring R^0 be as in Remark 3.7.*

- *If Φ is relevant, then R^0 generates $K(Y)$ as a field, or, in other words, the map of fields with kernel $\Phi^*: S(Y) \dashrightarrow \overline{S(X)}$ restricts to a map of fields with kernel $\Phi^*: K(Y) \dashrightarrow \overline{S(X)}$.*
- *If Φ satisfies both homogeneity and relevance conditions, then Φ^* restricts to a map of fields with kernel $\Phi^*: K(Y) \dashrightarrow K(X)$.*

Proof. Take a homogeneous regular section $f \in B_Y$ such that $f \notin \ker \Phi^*$. Let $\bar{U}_f = \bar{Y} \setminus \{f = 0\}$, i.e. $\bar{U}_f = \operatorname{Spec} S[Y]_{(f)}$ is an open affine G_Y -invariant subset of $\bar{Y} = \operatorname{Spec} S[Y]$. Moreover, \bar{U}_f is contained in the characteristic space $\hat{Y} = \bar{Y} \setminus Z(B_Y)$. The geometric invariant theory (GIT) guarantees the image $U_f = \pi_Y(\bar{U}_f)$ is an open affine subset of Y . Indeed, it is open by [Świe04, Cor. 3.7]. It is affine since $\bar{U}_f \rightarrow U_f$ is a good quotient and thus

$$U_f = \operatorname{Spec} \mathcal{O}_{\bar{X}}(\bar{U}_f)^{G_Y} = \operatorname{Spec} S[Y]_{(f)}.$$

Here $S[Y]_{(f)}$ is the homogeneous localisation of $S[Y]$ in f , that is the ring of degree 0 quotients $\frac{g}{f^d}$ for some homogeneous $g \in S[Y]$, and nonnegative integer d such that $\deg g = d \cdot \deg f$. The first part of the lemma follows from the fact that $R^0 = (S[Y]_{\ker \Phi^*})^0$ and that the latter contains $S[Y]_{(f)}$ which generates the field $K(Y)$.

The equivalent phrasing in terms of maps of fields with kernel follows immediately from Lemma 2.5. The second item follows from the homogeneity condition (A2). \square

6 Description of maps

In this section we exploit the language introduced so far to explain when a multi-valued map $\overline{X} \multimap \overline{Y}$ describes a rational map $X \dashrightarrow Y$. We characterise those multi-valued maps that describe some rational map in terms of the homogeneity and relevance conditions. Finally, we prove that every rational map between Mori Dream Spaces has a description.

Let $\Phi: \overline{X} \multimap \overline{Y}$ be a multi-valued map. Define a subset $\text{Reg}_Y \Phi \subset \text{Reg} \Phi$, the locus where $\pi_Y \circ \Phi$ is a well-defined map of sets:

$$\text{Reg}_Y \Phi := \{\xi \in \text{Reg} \Phi \mid \#\pi_Y(\Phi(\xi)) = 1\}.$$

This locus $\text{Reg}_Y \Phi$ may be empty. On the other hand, if $\text{Reg}_Y \Phi$ contains a nonempty open subset, then we regard Φ as being adapted to Y ; under this assumption, it makes sense to ask where Φ *agrees* with a rational map $X \dashrightarrow Y$.

Definition 6.1. Given a multi-valued map $\Phi: \overline{X} \multimap \overline{Y}$ and a rational map $\varphi: X \dashrightarrow Y$, in the notation above, the *agreement locus* of Φ and φ is

$$\text{Agr}(\Phi, \varphi) = \{\xi \in \text{Reg}_Y \Phi \cap \pi_X^{-1}(\text{Reg} \varphi) \mid \pi_Y \circ \Phi(\xi) = \varphi \circ \pi_X(\xi)\}.$$

In other words, the agreement locus is the set of points where both compositions $\pi_Y \circ \Phi$ and $\varphi \circ \pi_X$ are well-defined as maps of sets and they have the same values.

Definition 6.2. We say Φ *is a description of* φ , or that Φ *describes* φ , if $\text{Agr}(\Phi, \varphi)$ contains an open dense subset of \overline{X} .

Note that the definition of a description is a point-wise definition. However, it has strong algebraic consequences. Firstly, a description uniquely determines the map φ as in Remark 6.3. Secondly, a multi-valued map is a description if and only if it is homogeneous and relevant in the sense of Sections 4 and 5, see Proposition 6.4 and Theorem 6.5.

Remark 6.3. Rational maps are determined by the underlying maps of sets. More precisely:

- Suppose X and Y are algebraic varieties over an algebraically closed field of any characteristics. Suppose further $\varphi: X \dashrightarrow Y$ and $\psi: X \dashrightarrow Y$ are two rational maps and $U \subset X$ is an open dense subset, such that both $\varphi|_U$ and $\psi|_U$ are regular. If φ and ψ agree on U as maps of sets, i.e. $\forall x \in U$ we have $\varphi(x) = \psi(x)$, then $\varphi = \psi$ as rational maps.

- Back in our situation, if X and Y are Mori Dream Spaces over \mathbb{k} , if a multivalued map Φ describes two rational maps φ and ψ , then $\varphi = \psi$.

In the first bullet, compared with [Hart77, Lem. 4.1 and Def. on page 24], we are not assuming that the two morphisms, we only assume they agree as maps of sets.

Proof. Second item follows immediately from the definition and the first item.

The first item is standard: It suffices to show it for $X = U$ and φ, ψ regular maps. Then it suffices to prove it locally, i.e. assume X and Y are affine. Next assume Y is an affine space by composing our maps with a closed embedding. Finally, consider the problem coordinate by coordinate, i.e. assume $Y \simeq \mathbb{A}^1$. Then this is a problem about polynomial functions on an affine variety X , and the solution is straightforward. \square

One direction of the equivalence that Φ is a description if and only if it is homogeneous and relevant is straightforward (see also [BB13, Prop. 4.9]):

Proposition 6.4. *If Φ is a description of a rational map $\varphi: X \dashrightarrow Y$, then Φ satisfies the homogeneity and relevance conditions.*

Proof. By Definition 6.2 of description, $\pi_Y \circ \Phi$ is defined on an open subset of \overline{X} , so $\Phi(x)$ cannot be contained in the irrelevant locus for those points. Therefore Φ satisfies the relevance condition (B1).

Since Φ is a description the agreement locus $\text{Agr}(\Phi, \varphi)$ contains an open dense subset of $\text{Reg } \Phi$. The homogeneity condition (A4') is satisfied on this set. \square

The converse implication is slightly more involved (compare to [BB13, Thm 4.10]).

Theorem 6.5. *Let $\Phi: \overline{X} \rightarrow \overline{Y}$ be a multi-valued map that satisfies the homogeneity and relevance conditions. Then by its action on rational functions, Φ^* naturally determines a rational map $\varphi: X \dashrightarrow Y$, and Φ describes φ .*

Proof. By Lemma 5.2 the multivalued map Φ determines a map of fields with kernel $\Phi^*: K(Y) \dashrightarrow K(X)$. Let $R^0 \subset K(Y)$ be the ring generating $K(Y)$ from the definition of map of fields with kernel (Definition 2.4). By [BB13, Prop. 2.14(ii)] the ring homomorphism

$$\Phi^*: R^0 \rightarrow K(X)$$

determines a rational map $\varphi: X \dashrightarrow Y$ which is characterised by its action on rational functions $q \in K(Y)$ being $\varphi^*(q) = \Phi^*(q)$.

Next we have to prove that Φ describes φ . Consider the open subset¹

$$U = \text{Reg } \Phi \setminus \left(\text{Irrel}(X) \cup \Phi^{-1}(\text{Irrel}(Y)) \right).$$

Note that U is indeed open in \overline{X} or in $\text{Reg } \Phi$ as it is a complement of closed subsets: $\text{Irrel}(X)$ and $\text{Irrel}(Y)$ are closed by definition and $\Phi^{-1}(\text{Irrel}(Y))$ is closed by Proposition 3.10. Furthermore, if $\xi \in \Phi^{-1}(\text{Irrel}(Y))$, or equivalently if $\Phi(\xi) \cap \text{Irrel}(Y) \neq \emptyset$, then $\Phi(\xi) \subset \text{Irrel}(Y)$ by (A4). So the relevance condition (B1) guarantees that $\Phi^{-1}(\text{Irrel}(Y)) \neq \text{Reg } \Phi$ and U is non-empty. Since \overline{X} is irreducible (see [ADHL15, Thm 1.5.1.1]), the open subset is dense. Choose any $\xi \in U$. By the homogeneity condition (A4), $\pi_Y(\Phi(\xi))$ is a single point y . We claim $y = \varphi(\pi_X(\xi))$, so that $\xi \in \text{Agr}(\Phi, \varphi)$.

To prove the claim, we set $x = \pi_X(\xi)$ and evaluate rational functions $q \in K(Y)$ at $\varphi(x)$ and y :

$$q(\varphi(x)) = (\varphi^* q)(x) = (\Phi^* q)(\pi_X(\xi)) = q([\Phi(\xi)]) = q(y).$$

So no rational function on Y can distinguish between $\varphi(x)$ and y and therefore $y = \varphi(x)$. Hence $U \subset \text{Agr}(\Phi, \varphi)$ and Φ describes φ . \square

Corollary 6.6. *Suppose $\varphi: X \dashrightarrow Y$ is a rational map of Mori Dream Spaces, and $\Phi: \overline{X} \multimap \overline{Y}$ is a homogeneous and relevant multivalued map. Then the following conditions are equivalent:*

- (i) Φ describes φ .
- (ii) Φ^* and φ^* determine the same map of fields with kernel $S(Y)^0 = K(Y) \dashrightarrow S(X)^0 = K(X)$.

Proof. By Theorem 6.5 and its proof the multivalued map Φ describes some rational map $\psi: X \dashrightarrow Y$ and ψ is characterised by the property that $\psi^*: K(Y) \dashrightarrow K(X)$ is equal to $\Phi^*: S(Y)^0 = K(Y) \dashrightarrow S(X)^0$.

If (i) holds, then $\psi = \varphi$ by Remark 6.3 and (ii) is satisfied. Instead, assuming (ii), maps of fields with kernels φ^* and ψ^* are equal, thus also the rational maps φ and ψ are equal and (i) holds. \square

Now we show that every rational map has a description (see [BB13, Thm 4.12])

¹Our choice of open subset is anticipated from Proposition 7.1.

Theorem 6.7. *Let $\varphi: X \dashrightarrow Y$ be a rational map of Mori Dream Spaces. Then there exists a description $\Phi: \overline{X} \dashrightarrow \overline{Y}$ of φ . Moreover, Φ may be chosen in such a way that it satisfies the following zeroes condition:*

(Z) *for all homogeneous regular sections $f \in S[Y]$:*

$$\Phi^* f = 0 \iff \varphi(X) \subset \text{Supp}(f)$$

where $\text{Supp}(f) \subset Y$ denotes the support of the divisor in Y defined by f .

Proof. We must define the ring homomorphism $\Phi^*: S[Y] \rightarrow \overline{S(X)}$. We construct it in several steps. The construction will lead to a homomorphism from a significantly larger subring $R \subset S(Y)$, and in fact we are rather building a map of fields with kernel $\Phi^*: S(Y) \dashrightarrow \overline{S(X)}$ in the sense of §2.2.

Note that Φ^* (considered as a map of fields with kernel) after restriction to $K(Y) \subset S(Y)$ must coincide with φ^* by Corollary 6.6. Moreover, let $I = I(\varphi(X)) \subset S[Y]$ be the homogeneous ideal of (the closure of) the image of φ . We define the image of $f \in I$ under Φ^* to be 0. Thus it remains to define a homomorphism $S[Y]/I \rightarrow \overline{S(X)}$ keeping in mind the constraints determined by φ .

We will define the homomorphism $S[Y]/I \rightarrow \overline{S(X)}$ as a map on homogeneous elements of $S[Y]/I$ in such a way that the multiplicative structure is preserved and that the map is linear on each degree. These conditions are enough to extend the map to the required ring homomorphism. Since X is irreducible, and thus also its image $\varphi(X)$ is irreducible. Thus I is G_Y -prime, or equivalently the homogeneous elements of $S[Y]/I$ are not zero divisors. Therefore (as our interest are restricted to homogeneous elements), we are going to work with $S[Y]/I$ as if it was a domain, in particular we will consider fractions of homogeneous elements.

Let $G \subset \text{Cl}(Y)$ be the weight group of $S[Y]/I$, that is the group generated by the degrees d such that $(S[Y]/I)^d \neq 0$. Pick a finite sequence f_1, \dots, f_k of homogeneous sections $f_i \in S[Y]$, such that $\deg f_1, \dots, \deg f_k$ generate G and $f_i \notin I$. Define subgroups $A_i = \langle \deg f_1, \dots, \deg f_i \rangle \subset G$, and define the ascending sequence of subrings:

$$\mathbb{k} = T_0 \subset T_1 \subset T_2 \subset \dots \subset T_k = S[Y]/I,$$

where T_i comprises all gradings of $S[Y]/I$ in A_i . That is $T_i = \bigoplus_{a \in A_i} (S[Y]/I)^a$.

We will gradually extend our \mathbb{k} -algebra homomorphism $\overline{\Phi}^*: T_i \rightarrow \overline{S(X)}$ in such a way that for every $a \in A_i$:

- (i) $\frac{(T_i)^a}{S(X)}$ is mapped into the set of homogeneous multivalued sections in
- (ii) if $g \in (T_i)^a$ is mapped to 0, then $g = 0$.
- (iii) if $f, g \in (S[Y])^a$ and $g \notin I$, and $[f], [g] \in S[Y]/I$ are classes of f and g respectively, then $\frac{\overline{\Phi^*}([f])}{\overline{\Phi^*}([g])} = \varphi^* \left(\frac{f}{g} \right)$.

For $i = 0$, the homomorphism is defined and we proceed by induction on i . Suppose it is defined on T_i for some $0 \leq i < k$. We want to extend it to T_{i+1} . The class $[\deg f_{i+1}]$ generates the quotient group A_{i+1}/A_i . If the group $A_{i+1}/A_i \simeq \mathbb{Z}$, then set for instance $\overline{\Phi^*}[f_{i+1}] := 1$, or any other non-zero homogeneous multivalued section in $\overline{S(X)}$. If A_{i+1}/A_i is finite of order r instead, then $[f_{i+1}]^r \in T_i$, and set $\overline{\Phi^*}[f_{i+1}] := \sqrt[r]{[f_{i+1}]^r}$. Also in this case the image of $[f_{i+1}]$ is a non-zero homogeneous multivalued section.

Now pick any homogeneous section $g \in S[Y]^{\alpha \deg f_{i+1} + b}$ for some $b \in A_i$ and an integer α . Then $g \cdot f_{i+1}^{-\alpha} \in S(Y)$ has degree b , hence by Lemma 2.2 this homogeneous rational section is in the subfield of $S(Y)$ generated by $S(Y)^0$ and f_1, \dots, f_i . In other words, express b as an integral combination of the generators of A_i , $b = \sum_{j=1}^i \beta_j \deg f_j$. Then

$$q := g \cdot f_{i+1}^{-\alpha} \cdot f_1^{-\beta_1} \cdots f_i^{-\beta_i} \in S(Y)^0 = K(Y).$$

Note the poles of q are all contained in the union of zeroes of f_i . Since $f_i \notin I$ (by our choice of f_i), it makes sense to pullback q by φ . Hence we define:

$$\overline{\Phi^*}([g]) := \varphi^* q \cdot (\Phi^*[f_{i+1}])^\alpha \cdot \overline{\Phi^*}([f_1]^{\beta_1} \cdots [f_i]^{\beta_i}) \quad (6.8)$$

It is straightforward to check that the definition is linear in $g \in S[Y]^{\alpha \deg f_i + b}$ and equal to 0 for $g \in I$. So it does not depend on the choice of g, g' in the same equivalence class $[g] = [g'] \in S[Y]/I$. Moreover, if $g \notin I$, then $\overline{\Phi^*}([g])$ is a homogeneous multivalued section (cf. (i)) and $\overline{\Phi^*}([g]) \neq 0$ (cf. (ii)). Also by the inductive property (iii) the definition in (6.8) does not depend on the choice of integer α , or on the choice of expression $b = \sum_{j=1}^i \beta_j \deg f_j$. Moreover, the multiplicative structure is preserved by $\overline{\Phi^*}$ for homogeneous elements. Thus by linearity we extend $\overline{\Phi^*}$ to a ring homomorphism $T_{i+1} \rightarrow \overline{S(X)}$, which satisfies properties (i)–(iii).

Therefore inductively we have constructed a ring homomorphism

$$\overline{\Phi^*}: S[Y]/I \rightarrow \overline{S(X)},$$

and by composition with the quotient, also a desired ring homomorphism $\Phi^*: S[Y] \rightarrow \overline{S(X)}$. Note that (i) implies that Φ^* determines a multivalued

map $\Phi: \overline{X} \dashrightarrow \overline{Y}$ and Φ satisfies homogeneity condition (A1). Moreover, (ii) implies that condition (Z) holds, and hence also relevance (B2) holds.

Thus by Theorem 6.5 the multivalued map is a description of a map ψ . We conclude the proof using property (iii) and Corollary 6.6 which implies $\varphi = \psi$.

□

7 The agreement and disagreement loci

We describe the agreement locus of a description and then we prove its complement has only components of codimension 1 and components where the rational map is not regular. The content of this section is parallel to [BB13, §4.4].

Proposition 7.1. *Let Φ be a description of φ . Then*

$$\text{Agr}(\Phi, \varphi) = \text{Reg } \Phi \setminus \left(\text{Irrel}(X) \cup \Phi^{-1}(\text{Irrel}(Y)) \right).$$

Proof. By the definition of the agreement locus, if $\xi \in \text{Agr}(\Phi, \varphi)$, then

$$\xi \in \text{Reg } \Phi \setminus \text{Irrel}(X).$$

The homogeneity condition holds for Φ by Proposition 6.4, so, for such ξ , $\Phi(\xi)$ is contained in a single orbit by condition (A4) of Proposition 4.1. Since $\pi_Y(\Phi(\xi))$ is defined it follows that no point in $\Phi(\xi)$ is in $\text{Irrel}(Y)$, which proves the first inclusion:

$$\text{Agr}(\Phi, \varphi) \subset \text{Reg } \Phi \setminus \left(\text{Irrel}(X) \cup \Phi^{-1}(\text{Irrel}(Y)) \right).$$

To prove the other inclusion, take $\xi \in \text{Reg } \Phi \setminus \left(\text{Irrel}(X) \cup \Phi^{-1}(\text{Irrel}(Y)) \right)$ and set $y = \pi_Y(\Phi(\xi)) \in Y$. We must prove, that $x = \pi_X(\xi) \in \text{Reg } \varphi$ and that $\varphi(x) = y$, in other words that φ^* maps the local ring $\mathcal{O}_{Y,y} \subset K(Y)$ into the local ring $\mathcal{O}_{X,x} \subset K(X)$. So take any $q \in \mathcal{O}_{Y,y}$. By Corollary 6.6,

$$\varphi^* q = \Phi^* q \quad \text{as elements of } K(X).$$

Since a lift of y to \overline{Y} is in the image of Φ , it follows that $\Phi^* q$ is defined and hence $\varphi^* q$ is defined. Hence we can calculate:

$$(\varphi^* q)(x) = (\Phi^* q)(\xi) = q(\Phi(\xi)) = q(y),$$

where the outer equalities hold because rational functions can be evaluated on any representative of a point in the Cox cover. Since q is regular at y , also $\varphi^*q \in \mathcal{O}_{X,x}$ as claimed. So $\varphi(x) = y$ and thus $\xi \in \text{Agr}(\Phi, \varphi)$. \square

Corollary 7.2. *Let Φ be a description of φ . The agreement locus $\text{Agr}(\Phi, \varphi)$ is an open G_X -invariant subset of \overline{X} (and of $\text{Reg } \Phi$). $\pi_X(\text{Agr}(\Phi, \varphi))$ contains an open dense subset of X .*

Proof. $\text{Reg } \Phi$ is an open G_X -invariant subset by Corollary 3.5. $\text{Irrel}(X)$ is clearly closed and G_X -invariant. Finally, $\text{Irrel}(Y)$ is a G_Y -invariant subset of \overline{Y} , so by homogeneity condition (A4) also $\Phi^{-1}(\text{Irrel}(Y))$ is G_X -invariant, and it is closed in $\text{Reg } \Phi$ by Proposition 3.10. Thus $\text{Agr}(\Phi, \varphi)$ is open and G_X -invariant by Proposition 7.1. \square

The definition of the agreement locus gives $\text{Agr}(\Phi, \varphi) \subset \pi_X^{-1}(\text{Reg } \varphi)$. In §8, we distinguish those descriptions for which the equality holds and we prove that they always exist. In the meantime, we call the difference between the two sets the *disagreement locus*.

Proposition 7.3. *Let $\varphi: X \dashrightarrow Y$ be a rational map between two toric varieties X and Y with a description $\Phi: \overline{X} \dashrightarrow \overline{Y}$. Consider two open subsets $U_2 \subset U_1$ of \overline{X} :*

$$U_1 = \pi_X^{-1}(\text{Reg } \varphi) \quad \text{and} \quad U_2 = \text{Agr}(\Phi, \varphi).$$

The disagreement locus $D = U_1 \setminus U_2$ is a closed subset in U_1 purely of codimension 1 in U_1 or is empty.

Proof. Since U_2 is a non-empty open subset of U_1 by Corollary 7.2, clearly D is a proper closed subset in U_1 . By Proposition 7.1 we have an equality

$$U_2 = \text{Reg } \Phi \setminus \left(\text{Irrel}(X) \cup \Phi^{-1}(\text{Irrel}(Y)) \right).$$

Note that $\text{Irrel}(X)$ is disjoint from U_1 (because π_X is not regular on $\text{Irrel}(X)$). Therefore

$$D = \underbrace{\left(U_1 \setminus \text{Reg } \Phi \right)}_{=: D_{\text{ind}}} \cup \underbrace{\left(U_1 \cap \Phi^{-1}(\text{Irrel}(Y)) \right)}_{=: D_{\text{irrel}}}.$$

By Corollary 3.5 the locus D_{ind} is indeed purely of codimension 1 (or empty). It therefore remains to prove that also D_{irrel} is purely of codimension 1 or empty.

Assume D_{irrel} is not empty and choose arbitrary $\xi \in D_{\text{irrel}}$. We have to prove the codimension of D_{irrel} at ξ is 1. Since $\xi \in U_1$ the rational map φ is regular at $x = \pi_X(\xi)$. Consider $y = \varphi(x)$ and its open affine neighbourhood $V \subset Y$, such that V is given by non-vanishing of certain homogeneous regular section $g \in S[Y]$, say $V = \{g \neq 0\}$. Set $\gamma = \Phi^*g$. By homogeneity condition (A1), there exists $f \in \mathbb{k}[\text{Reg } \Phi]$ such that $\gamma^r = f$ for some $r \geq 1$. We claim that $f(\xi) = 0$ and that for all ξ' in the locus $\{f = 0\}$ and in some sufficiently small open neighbourhood of ξ we have $\xi' \in D_{\text{irrel}}$.

First we prove $f(\xi) = 0$. Since $\xi \in \Phi^{-1}(\text{Irrel}(Y))$ it follows $\Phi(\xi)$ and $\text{Irrel}(Y)$ have non-empty intersection. As usual, since $\Phi(\xi)$ is contained in a single torus orbit by the homogeneity condition (A4), we have $\Phi(\xi) \subset \text{Irrel}(Y)$. In particular, $\Phi(\xi)$ is disjoint from $\pi_Y^{-1}(V)$, in other words the section g vanishes on $\Phi(\xi)$. So γ vanishes at ξ and therefore f vanishes on ξ .

We prove further that $\xi' \in \{f = 0\}$ implies $\xi' \in D_{\text{irrel}}$, at least in some neighbourhood of ξ . More precisely, we take this neighbourhood to be

$$(\varphi \circ \pi_X)^{-1}(V) \cap \text{Reg } \Phi.$$

Since Φ is regular at such ξ' :

$$0 = f(\xi') = \gamma^r(\xi') = \Phi^*(g^r)(\xi') = g^r(\Phi(\xi')),$$

so $\Phi(\xi')$ is contained in the locus $g = 0$. Therefore $\Phi(\xi')$ is disjoint from $\pi_Y^{-1}(V)$ and hence the set $\pi_Y(\Phi(\xi'))$ (if non-empty) is not in V . On the other hand $\varphi(\xi')$ is contained in V by our choice of the open neighbourhood of ξ . We conclude, that ξ' cannot be in the agreement locus U_2 . But $\xi' \in \text{Reg } \Phi$ and $\xi' \notin \text{Irrel}(X)$ (again by our choice of open neighbourhood of ξ). Therefore by Proposition 7.1 there is no other possibility than $\xi' \in \Phi^{-1}(\text{Irrel}(Y))$ so that $\xi' \in D_{\text{irrel}}$ as claimed.

Hence D_{irrel} locally near ξ contains a subset $\{f = 0\}$ purely of codimension 1. Since the same holds true for every $\xi \in D_{\text{irrel}}$ and $D_{\text{irrel}} \neq U_1$, we conclude that D_{irrel} is purely of codimension 1. \square

8 Existence of complete descriptions

Definition 8.1. A description Φ of $\varphi: X \dashrightarrow Y$ is *complete* if it satisfies

$$(C) \text{ Agr}(\Phi, \varphi) = \pi_X^{-1}(\text{Reg } \varphi).$$

Proposition 7.1, together with this definition, has an immediate corollary.

Corollary 8.2. *If Φ is a complete description of φ , then*

$$\text{Reg } \varphi = \pi_X(\text{Reg } \Phi \setminus \Phi^{-1}(\text{Irrel}(Y))).$$

In particular, the map φ is regular on X if and only if Φ is regular on \overline{X} and $\Phi^{-1}(\text{Irrel}(Y))$ is contained in $\text{Irrel}(X)$.

The main claim of this article is that complete descriptions of maps between Mori Dream Spaces always exist.

Theorem 8.3. *Let $\varphi: X \dashrightarrow Y$ be a rational map of Mori Dream Spaces. Then there exists a complete description $\Phi: \overline{X} \dashrightarrow \overline{Y}$ of φ . Moreover, Φ may be chosen in such a way that it satisfies zeroes condition (Z) of Theorem 6.7.*

The theorem is proved throughout this section. We commence with an additional assumption that Y is a toric variety. For this part, we follow the strategy of the arguments of [BB13, §4.5], where both X and Y are assumed to be toric varieties. Later, to conclude we use the embedding theorem for Mori Dream Spaces. Every Mori Dream Space admits a natural closed embedding into a toric variety, see §8.2 for details and references.

8.1 Case when the target is a toric variety

Suppose Y is a normal toric variety, with Cox ring $S[Y] \simeq \mathbb{k}[y_1, \dots, y_n]$, where the generators y_i are homogeneous of degrees $\deg(y_i) \in \text{Cl}(Y)$, and they correspond to rays of the fan Σ_Y (see [CLS11, §5.1]).

The following lemma is clear.

Lemma 8.4. *Suppose $\eta \in \overline{Y} \simeq \mathbb{k}^n$ is a closed point $\eta = (\eta_1, \dots, \eta_n)$. Pick a group homomorphism $w: \text{Cl}(Y) \rightarrow \mathbb{Q}$ and any $t \in \mathbb{k}^*$. Then the point $t^w \cdot \eta := (t^{w(\deg y_1)} \eta_1, \dots, t^{w(\deg y_n)} \eta_n)$ is in the same G_Y -orbit as η .*

□

Now we add to the picture a Mori Dream Space X and a rational map $\varphi: X \dashrightarrow Y$. By Theorem 6.7 there exists a description

$$\Phi: \overline{X} \dashrightarrow \overline{Y} \simeq \mathbb{k}^n$$

satisfying zeroes condition (Z). What are the possibilities to modify Φ , so that it still describes φ and satisfies (Z)?

If $f \in S[X]$ is a homogeneous regular section and $w: \text{Cl}(Y) \rightarrow \mathbb{Q}$ is a group homomorphism, then we define a multi-valued map

$$\begin{aligned} f^w \cdot \Phi: \overline{X} &\dashrightarrow \overline{Y} \\ \xi &\mapsto f(\xi)^w \cdot \Phi(\xi) = (f^{w(\deg y_1)} \Phi^* y_1, \dots, f^{w(\deg y_n)} \Phi^* y_n). \end{aligned}$$

Corollary 8.5. $f^w \cdot \Phi$ describes φ and their agreement loci are the same away from $\{f = 0\}$:

$$\text{Agr}(\Phi, \varphi) \cap \{f \neq 0\} = \text{Agr}(f^w \cdot \Phi, \varphi) \cap \{f \neq 0\}.$$

Moreover, $f^w \cdot \Phi$ satisfies (Z).

Proof. On a sufficiently general point ξ of \overline{X} (such that $f(\xi) \neq 0$) the image $(f^w \cdot \Phi)(\xi)$ is in the same G_Y -orbit as the image of $\Phi(\xi)$ by Lemma 8.4. Thus $f^w \cdot \Phi$ describes φ and the agreement locus is as claimed. Furthermore, say $g \in S[Y]$ is a homogeneous section of degree $\deg g$. Then

$$(f^w \cdot \Phi)^* g = f^{w(\deg g)} \Phi^* g$$

and since $f^{w(\deg g)}$ is invertible in $\overline{S(\overline{X})}$ we have

$$(f^w \cdot \Phi)^* g = 0 \iff \Phi^* g = 0 \iff \varphi(X) \subset \text{Supp}(g).$$

Thus the condition (Z) is satisfied. \square

We want to consider the toric variety $Z \subset Y$, which is the smallest (closed) toric stratum of Y , which contains $\varphi(X)$.

Let $\sigma \in \Sigma_Y \subset N \otimes \mathbb{R}$ be the cone corresponding to Z , and let $\Sigma_{Y,Z} = \{\tau_Y \in \Sigma_Y \mid \sigma \preceq \tau_Y\}$ be the subfan of Σ_Y corresponding to the smallest torus invariant open neighbourhood in Y of the toric stratum Z . Let $N_\sigma = \sigma \cap N$ be the sublattice of N generated by σ and let $N(\sigma) = N/N_\sigma$ be the quotient lattice. Let $\Sigma_Z = \text{Star}(\sigma)$ be the fan given by the images of cones in $\Sigma_{Y,Z}$ under the natural projection $p: N \rightarrow N(\sigma)$; this is the fan of Z regarded as a toric variety under the $T_{N(\sigma)}$ torus action, cf. [CLS11, §3.2] (If $\varphi(X)$ is not contained in any strict toric stratum of Y , then $Z = Y$ and Σ_Z is equal to Σ_Y).

Denote the primitive generators of rays of Σ_Y by ρ_1, \dots, ρ_n , where the variable y_i in the Cox ring $S[Y]$ corresponds to the ray spanned by ρ_i . Let $\hat{N} = \bigoplus_{i=1}^n \mathbb{Z} e_{\rho_i}$ be the group of torus invariant Weil divisors on Y . Define L to be the natural surjective linear map $L: \hat{N} \rightarrow N(\sigma)$ sending e_{ρ_i} to $p(\rho_i)$ and set $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$. The kernel of $L_{\mathbb{Q}}$ is responsible for the freedom we have in modifying the descriptions.

Lemma 8.6. Suppose $f \in S[X]$ is a non-zero homogeneous section and $\mu = (\mu_1, \dots, \mu_n) \in \ker L_{\mathbb{Q}}$. Then the multi-valued map $\Psi: \overline{X} \rightrightarrows \overline{Y}$ given by

$$\Psi^* y_i := f^{\mu_i} \Phi^* y_i.$$

is also a description of φ . Moreover Ψ satisfies zeroes condition (Z) and

$$\text{Agr}(\Phi, \varphi) \cap \{f \neq 0\} = \text{Agr}(\Psi, \varphi) \cap \{f \neq 0\}.$$

Proof. Set $R = \{i \mid \rho_i \in \sigma(1)\}$ and $R' = \{1, \dots, n\} \setminus R$. The zeroes condition (Z) and the choice of $Z \subset Y$ assure that $\Phi^*y_i = 0$ if and only if $i \in R$. The vector space $N_\sigma \otimes \mathbb{Q}$ is spanned by $\{\rho_i : i \in R\}$ and, by the assumption, $\sum_{j \in R'} \mu_j \rho_j \in N_\sigma \otimes \mathbb{Q}$. Choose $\nu_i \in \mathbb{Q}$ such that the latter is equal to $\sum_{i \in R} \nu_i \rho_i$. Set $\mu' = \sum_{j \in R'} \mu_j e_{\rho_j} - \sum_{i \in R} \nu_i e_{\rho_i}$ and $\mu'' = \sum_{i \in R} (\nu_i + \mu_i) e_{\rho_i}$. Then $\mu = \mu' + \mu''$ and μ' defines a homomorphism $w: \text{Cl}(Y) \rightarrow \mathbb{Q}$ by setting $w(\deg y_i) = -\nu_i$ for $i \in R$ and $w(\deg y_j) = \mu_j$ for $j \in R'$.

Note $f^{\mu_i} \Phi^*y_i = f^{-\nu_i} \Phi^*y_i = 0$ for $i \in R$. Therefore

$$f^{\mu_i} \Phi^*y_i = f^{w(e_i)} \Phi^*y_i,$$

or equivalently, $\Psi = f^w \cdot \Phi$ and the claim follows from Corollary 8.5 \square

Fix a prime divisor $(f) \subset X$ (in particular, $f \in S[X]$ is homogeneous and G_X -irreducible). For all $i \in \{1, \dots, n\}$ let μ_i be the multiplicity of f in Φ^*y_i if the pullback is non-zero, or pick any number $\mu_i \in \mathbb{Q}$, if $\Phi^*y_i = 0$. Set $\mu = \sum_{i=1}^n \mu_i e_{\rho_i} \in \widehat{N} \otimes \mathbb{Q}$.

Lemma 8.7. *Let $m \in \text{Hom}_{\mathbb{Z}}(N(\sigma), \mathbb{Z})$ be an integral linear form corresponding to the rational function χ^m on Z . Then, for $f \in S[X]$ as above, $L_{\mathbb{Q}}(\mu)$ is an integral point of $N(\sigma)$ and the order of vanishing of $\varphi^* \chi^m$ along the divisor (f) is equal to $m \circ L(\mu)$.*

Proof. The composition of $m \circ p$ is an integral form on the lattice N and hence $\chi^{m \circ p}$ is a rational function on Y whose order of vanishing along the toric divisor corresponding to the ray spanned by ρ_i is equal to $m \circ L(e_{\rho_i})$. The order of Φ^*y_i along (f) is by definition μ_i so the order of $\varphi^* \chi^m$ (which is equal to $\Phi^* \chi^{m \circ p}$ by Corollary 6.6) along (f) is $m \circ L_{\mathbb{Q}}(\mu)$. Since m is arbitrary $L_{\mathbb{Q}}(\mu) \in N(\sigma)$ is integral. \square

Corollary 8.8. *For $f \in S[X]$ as above, if $L_{\mathbb{Q}}(\mu)$ is not in the support of Σ_Z , then φ is not regular on (f) .*

Proof. Let τ be any cone in Σ_Z . Since $L(\mu) \notin \tau$, there exists $m_\tau \in \text{Hom}_{\mathbb{Z}}(N(\sigma), \mathbb{Z})$ in the dual cone τ^\vee such that $m_\tau \circ L(\mu) < 0$. Then, by Lemma 8.7, the rational function $\varphi^* \chi^{m_\tau}$ has a pole along (f) . Let U_τ be the affine open subset of Y corresponding to a cone in $\Sigma_{Y,Z}$, which maps onto τ . Note that the collection of such U_τ for all $\tau \in \Sigma_Z$ will cover the image of φ . Then $m_\tau \circ p$ is a regular function on U_τ which pullback is not regular on (f) . By [BB13, Prop. 2.15] this implies that φ is not regular on (f) . \square

We are now ready to prove:

Proposition 8.9. *Let X be a Mori Dream Space and Y a toric variety. Suppose $\varphi: X \dashrightarrow Y$ is a rational map. Then there exists a complete description $\Phi: \overline{X} \rightrightarrows \overline{Y}$ of φ . Moreover, Φ may be chosen in such a way it satisfies zeroes condition (Z) of Theorem 6.7.*

Proof. By Theorem 6.7 there exists a description Φ of φ satisfying (Z). We have to modify this description to obtain a complete one.

By Proposition 7.3, the disagreement locus

$$D = \pi_X^{-1}(\text{Reg } \varphi) \setminus \text{Agr}(\Phi, \varphi)$$

is a union of codimension 1 components. If D is empty, then the proposition is proved, so suppose it is not empty; we must modify Φ so that the new description is defined on those components which cover the locus where φ is defined.

Choose any homogeneously prime component of D and pick a G_X -irreducible section $f \in S[X]$ that vanishes to order 1 along this component. As above, for all $i \in \{1, \dots, n\}$ let μ_i be the multiplicity of f in Φ^*y_i if the pullback is non-zero, or pick any number $\mu_i \in \mathbb{Q}$, if $\Phi^*y_i = 0$.

By Corollary 8.8, if $L(\mu)$ does not lie in the support of Σ_Z , then (f) is not a part of the disagreement locus, contradicting our setup. Thus $L(\mu)$ lies in the support of Σ_Z .

Let τ be the cone in Σ_Z of minimal dimension that contains $L(\mu)$ and let τ_Y be a maximal cone in $\Sigma_{Y,Z}$ that maps onto τ . Pick a vector $u \in \tau_Y$ that maps to $L(\mu)$. There exists a vector $\mu' = (\mu'_1, \dots, \mu'_n) \in \widehat{N}$ with $\mu'_i > 0$ for $\rho_i \in \tau_Y(1)$ and $\mu'_i = 0$ otherwise, such that $p(\mu') = u$. Since $\mu' - \mu \in \ker L$, by Lemma 8.6, the multi-valued map Ψ

$$\Psi^*y_i := f^{\mu'_i - \mu_i} \cdot \Phi^*y_i.$$

is another description of φ with the same (dis)agreement locus as Φ away from $\{f = 0\}$. We claim that the component $\{f = 0\}$ is not in the disagreement of Ψ and φ .

By Proposition 7.1, it is enough to prove the following two statements:

- Ψ is regular on a general point of (f) .
- Ψ does not map a general point of (f) into the irrelevant locus of Y .

The first is immediate: Ψ is regular along (f) since the multiplicity of f in any non-zero Ψ^*y_i is equal to μ'_i which is non-negative. Moreover, this shows that if $x \in \{f = 0\}$ is a general point, then $\Psi(x)$ has zero y_i -coordinate if and only if either $\Phi^*y_i = 0$ or $\mu'_i > 0$. In particular, if $\rho_i \notin \tau_Y(1)$, then

the i -th coordinate of $\Psi(x)$ is non-zero. This means that the generator of B_Y determined by τ_Y is non-zero at $\Psi(x)$ so x is not mapped by Ψ into the irrelevant locus of Y . Therefore $\text{Agr}(\Psi, \varphi)$ contains a general point of $\{f = 0\}$ as claimed.

Thus we have obtained a description Ψ of φ whose disagreement locus contains one component less than that of Φ . Continuing inductively, we obtain a description with an empty disagreement locus, namely a complete description. \square

8.2 General case

Now let Y be a Mori Dream Space. Then there exists a toric variety Z and a closed embedding $\iota: Y \rightarrow Z$, such that $\text{Cl}(Z) \simeq \text{Cl}(Y)$ and the Cox rings satisfy the graded equality $S[Y] = S[Z]/I$, where $I = I(\iota(Z)) \subset S[Z]$ is the homogeneous ideal of sections vanishing on $\iota(Y)$, see [ADHL15, Thm 3.2.1.4, Construction 3.2.5.3, Prop. 3.2.5.4(i),(iii)]]. Using this embedding we can reduce the general case of Theorem 8.3 to the case when the target is a toric variety.

Proof of Theorem 8.3. Suppose X and Y are Mori Dream Spaces, and $\varphi: X \dashrightarrow Y$ is a rational map. Let Z be the toric variety and $\iota: Y \rightarrow Z$ be the closed embedding as above. Set $\psi := \iota \circ \varphi$ to be the composed rational map. By Proposition 8.9, there exists a complete description $\Psi: \overline{X} \dashrightarrow \overline{Z}$ of ψ , satisfying in addition zeroes condition (Z). In particular, for all sections $f \in I = I(\iota(Y))$, the pullback $\Psi^* f$ vanishes, because $\psi(X) \subset \iota(Y)$. Thus Ψ^* factorises through $S[Z]/I = S[Y]$:

$$S[Z] \rightarrow S[Z]/I = S[Y] \xrightarrow{\Phi^*} \overline{S(X)}.$$

We define the multi-valued map $\Phi: \overline{X} \dashrightarrow \overline{Y}$ by the algebra homomorphism Φ^* arising from this factorisation. To make sure this is well defined, we must check that on a distinguished set of homogeneous generators $y_i \in S[Y]$ the image $\Phi^* y_i$ is homogeneous. But homogeneity condition (A1) for Ψ guarantees in fact a stronger statement, that the same homogeneity condition holds for Φ . Note that here we exploit that $\text{Cl}(Y) = \text{Cl}(Z)$, and thus $G_Y = G_Z$.

We claim Φ is a complete description of φ . Firstly note that $\text{Reg } \Phi = \text{Reg } \Psi$, since the denominators of images of $\Phi^*: S[Y] \rightarrow \overline{S(X)}$ are the same as the denominators of the images of $\Psi^*: S[Z] \rightarrow \overline{S(X)}$ (simply, the two images are equal). Pick $\xi \in \widehat{X}$ such that φ is defined at $\pi_X(\xi)$. Then also $\psi = \iota \circ \varphi$ is defined at $\pi_X(\xi)$ and thus (since Ψ is a complete description) both

Ψ and Φ are regular at ξ . Moreover, $\Psi(\xi)$ is contained in a single G_Z -orbit, the sections of I (treated as functions on \overline{Z}) vanish on $\Psi(\xi)$, and the image $\pi_Z(\Psi(\xi))$ is a single point in Z equal to $\psi \circ \pi_X(\xi) = \iota \circ \varphi \circ \pi_X(\xi)$. Therefore the G_Z orbit containing $\Psi(\xi)$ is contained in $\overline{Y} \subset \overline{Z}$, and $\Phi(\xi)$ is in the same G_Y -orbit (again, recall $G_Y = G_Z$). Thus $\pi_Y(\Phi(\xi))$ is a single point and it is equal to $\varphi \circ \pi_X(\xi)$ as claimed. \square

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